Goodness of Fit Statistics for Poisson Regression
Outline

• Example 3: Recall of Stressful Events

• Goodness of fit statistics
  – Pearson Chi-Square test
  – Log-Likelihood Ratio test
Example 3: Recall of Stressful Events

• Let us explore another (simple) Poisson model example (no covariate to start with)
Example 3: Recall of Stressful Events

- Participants of a randomised study where asked if they had experienced any stressful events in the last 18 months. If yes, in which month?

- 147 stressful events reported in the 18 months prior to interview.
Example 3: Recall of Stressful Events

- $H_0$: Events uniformly distributed over time.

$H_0: p_1 = p_2 = ... = p_{18} = 1/18 = 0.055$

where $p_i$ = probability of event in month $i$.

i.e. we would expect about 5.5% of all events per month
## Example 3: Recall of Stressful Events Data

<table>
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<th>%</th>
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Evaluation of Poisson Model

• Let us evaluate the model using Goodness of Fit Statistics
  • Pearson Chi-square test
  • Deviance or Log Likelihood Ratio test for Poisson regression

• Both are goodness-of-fit test statistics which compare 2 models, where the larger model is the saturated model (which fits the data perfectly and explains all of the variability).
In this last example, if $H_0$ is true the expected number of stressful events in month $i$ (in any month) is (equiprobable model)

$$E(y_i) = m_i = 147 \times (1/18) = 8.17$$

$$\log(m_i) = a \quad i = 1, \cdots, C$$

i.e. we have a model with one parameter $a$. 
# Observed and expected count

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<td>18</td>
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Pearson Chi-Squared Test Statistic

- The **Pearson chi-squared test statistic** is the sum of the standardized residuals squared

\[
X^2 = \sum_{\text{cells}} \left[ \frac{O_i - E_i}{\sqrt{E_i}} \right]^2
\]

\[
= \left( \frac{15 - 8.17}{\sqrt{8.17}} \right)^2 + \left( \frac{11 - 8.17}{\sqrt{8.17}} \right)^2 + \ldots + \left( \frac{4 - 8.17}{\sqrt{8.17}} \right)^2 = 45.4
\]
Pearson Chi-Squared Test Statistic

• If $H_0$ is true
  
  $$X^2 \sim \chi^2_{df}$$

  where df = degrees of freedom
  
  = no. of cells – no. of model parameters
  
  = C - 1

• $X^2 = 45.4$ with 17 df (at 5% significance level the value from the chi-square table is 27.6)
  
  p-value < 0.001 → reject $H_0$.

• Conclusion: There is strong evidence that the equiprobable model does not fit the data.
Log Likelihood Ratio Test Statistic for Poisson Regression

• The Log Likelihood Ratio test statistic (also called Deviance of the Poisson Model) is

\[ L^2 = 2 \sum_{\text{cells } i} O_i \log \left( \frac{O_i}{E_i} \right) \]

• This can be used as a measure of the fit of the model (goodness of fit statistics)
Log Likelihood Ratio Test

• **If $H_0$ is true**

\[ L^2 \sim \chi^2_{df} \]

where $df = \text{degrees of freedom}

= \text{no. of cells} - \text{no. of model parameters}

= C - 1

• $L^2 = 50.8$ with 17 df.

  $p$-value $< 0.001 \rightarrow \text{reject } H_0.$

• **Conclusion:** There is strong evidence that the equiprobable model does not fit the data.
Remarks

• $X^2$ and $L^2$ are asymptotically equivalent. If they are not similar, this is an indication that the large sample approximation may not hold.

• For fixed df, as $n$ increases the distribution of $X^2$ usually converges to $\chi^2_{df}$ more quickly than $L^2$. The chi-squared approximation is usually poor when expected cell frequencies are less than 5.